

# Sigma clique covering of graphs

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## Abstract

The sigma clique cover number (resp. sigma clique partition number) of graph  $G$ , denoted by  $\text{scc}(G)$  (resp.  $\text{scp}(G)$ ), is defined as the smallest integer  $k$  for which there exists a collection of cliques of  $G$ , covering (resp. partitioning) all edges of  $G$  such that the sum of sizes of the cliques is at most  $k$ . In this paper, among some results we provide some tight bounds for  $\text{scc}$  and  $\text{scp}$ .

**Keywords:** clique covering; clique partition; sigma clique covering; sigma clique partition; set intersection representation; set system.

## 1 Introduction

Throughout the paper, all graphs are simple and undirected. By a *clique* of a graph  $G$ , we mean a subset of mutually adjacent vertices of  $G$  as well as its corresponding complete subgraph. The *size* of a clique is the number of its vertices. Also, a *biclique* of  $G$  is a complete bipartite subgraph of  $G$ . A *clique covering* (resp. *biclique covering*) of  $G$  is defined as a family of cliques (resp. bicliques) of  $G$  such that every edge of  $G$  lies in at least one of the cliques (resp. bicliques) comprising this family. A clique (resp. biclique) covering in which each edge belongs to exactly one clique (resp. biclique), is called a *clique* (resp. *biclique*) *partition*. The minimum size of a clique covering, a biclique covering, a clique partition and a biclique partition of  $G$  are called *clique cover number*, *biclique cover number*, *clique partition number* and *biclique partition number* of  $G$  and are denoted by  $\text{cc}(G)$ ,  $\text{bc}(G)$ ,  $\text{cp}(G)$  and  $\text{bp}(G)$ , respectively.

The subject of clique covering has been widely studied in recent decades. First time, Erdős et al. in [6] presented a close relationship between the clique covering and the set intersection representation. Also, they proved that the clique partition number of a graph on  $n$  vertices cannot exceed  $n^2/4$  (known as Erdős-Goodman-Pósa theorem). The connections of clique covering and other combinatorial objects have been explored (see e.g. [14, 16]). For a survey of the classical results on the clique and biclique coverings see [11, 13].

Chung et al. in [4] and independently Tuza in [15] considered a weighted version of the biclique covering. In fact, given a graph  $G$ , they were concerned with minimizing  $\sum_{B \in \mathcal{B}} |V(B)|$  among all biclique coverings  $\mathcal{B}$  of  $G$ . They proved that every graph on  $n$  vertices has a biclique covering such that the sum of number of vertices of these bicliques is  $O(n^2/\log n)$  [4, 15]. Furthermore, a clique counterpart of weighted biclique cover number has been studied. Following a conjecture by Katona and Tarjan, Chung [3], Gyori and Kostochka [7] and Kahn [10], independently, proved that every graph on  $n$  vertices has a clique partition such that the sum of number of vertices in these cliques is at most  $n^2/2$ . This can be considered as a generalization of Erdős-Goodman-Pósa theorem.

In this paper, we are concerned with a weighted version of the clique cover number. Let  $G$  be a graph. The *sigma clique cover number* of  $G$ , denoted by  $\text{scc}(G)$ , is defined as the minimum integer  $k$  for which there exists a clique covering  $\mathcal{C}$  of  $G$ , such that the sum of its clique sizes is at most  $k$ . For a clique covering  $\mathcal{C}$  of a graph  $G$  and a vertex  $u \in V(G)$ , let the *valency* of  $u$  (with respect to  $\mathcal{C}$ ), denoted by  $\mathcal{V}_{\mathcal{C}}(u)$ , be the number of cliques in  $\mathcal{C}$  containing  $u$ . In fact,

$$\text{scc}(G) = \min_{\mathcal{C}} \sum_{C \in \mathcal{C}} |C| = \min_{\mathcal{C}} \sum_{u \in V(G)} \mathcal{V}_{\mathcal{C}}(u),$$

where the minimum is taken over all clique coverings of  $G$ . Analogously, one can define *sigma clique partition number* of  $G$ , denoted by  $\text{scp}(G)$ . As a matter of fact, the above-mentioned result in [3, 7, 10] states that for every graph  $G$  on  $n$  vertices,  $\text{scp}(G) \leq n^2/2$ .

In order to reveal inherent difference between  $\text{cc}(G)$  and  $\text{scc}(G)$ , we introduce a similar parameter  $\text{scc}'(G)$  which is defined as the minimum of the sum of clique sizes in a clique covering  $\mathcal{C}$  achieving  $\text{cc}(G)$ , i.e.

$$\text{scc}'(G) := \min \left\{ \sum_{C \in \mathcal{C}} |C| : \mathcal{C} \text{ is a clique covering of } G \text{ and } |\mathcal{C}| = \text{cc}(G) \right\}.$$

It is evident that  $\text{scc}(G) \leq \text{scc}'(G)$ . In Section 2, first in Theorem 1, we will see that for some classes of graphs  $G$ , the quotient  $\text{scc}'(G)/\text{scc}(G)$  can be arbitrary large. Then, we give some general bounds on the sigma clique cover number and the sigma clique partition number. In particular, we prove that if  $G$  is a graph on  $n$  vertices with no isolated vertex and the maximum degree of the complement of  $G$  is  $d-1$ , for some integer  $d$ , then  $\text{scc}(G) \leq cnd \lceil \log((n-1)/(d-1)) \rceil$ , where  $c$  is a constant. We conjecture that this upper bound is best up to a constant factor for large enough  $n$ . In Section 3, using a well-known result by Bollobás, we prove the correctness of this conjecture for  $d=2$ . In other words, we show that for every even integer  $n$ , if  $G$  is the complement of an induced matching on  $n$  vertices, then  $\text{scc}(G) \sim n \log n$ . Finally, in Section 4 we give an interpretation of this conjecture as an interesting set system problem.

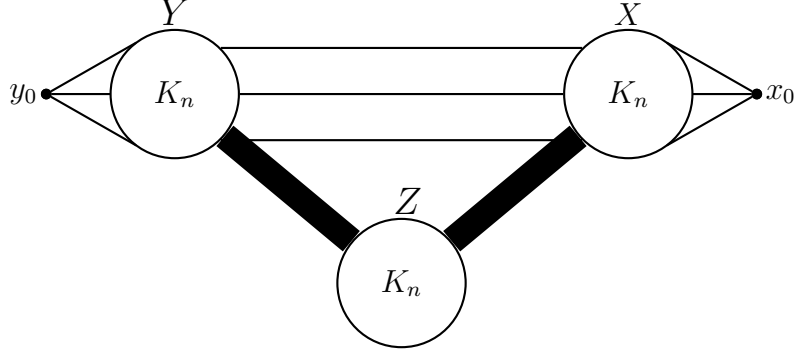


Figure 1: The graph  $G_n$ .

## 2 Some Bounds

In this section, first we present a class of graphs for which the family of clique coverings achieving  $\text{cc}(G)$  is disjoint from the family of clique coverings achieving  $\text{scc}(G)$ . Then, we provide several inequalities relating the introduced clique covering parameters. Moreover, we present an upper bound for  $\text{scc}(G)$  in terms of the number of vertices and the maximum degree of the complement of  $G$ .

**Theorem 1.** *There exists a sequence of graphs  $\{G_n\}$  such that  $\text{scc}'(G_n)/\text{scc}(G_n)$  tends to infinity as  $n$  tends to infinity.*

*Proof.* Let  $n$  be a positive integer and  $G_n$  be a graph on  $3n+2$  vertices, such that  $V(G_n) = \{x_0, y_0\} \cup X \cup Y \cup Z$ , where  $X = \{x_1, \dots, x_n\}$ ,  $Y = \{y_1, \dots, y_n\}$  and  $Z = \{z_1, \dots, z_n\}$  and adjacency is as follows. The sets  $X \cup \{x_0\}$ ,  $Y \cup \{y_0\}$  and  $Z$  are three cliques and every vertex in  $Z$  is adjacent to every vertex in  $X \cup Y$ . Moreover, for all  $i, j \in \{1, \dots, n\}$ ,  $x_i$  is adjacent to  $y_j$  if and only if  $i = j$  (see Figure 1).

First, note that each clique of  $G_n$  covers at most one edge from the set  $\{x_i y_i : 1 \leq i \leq n\} \cup \{x_0 x_1, y_0 y_1\}$ . This yields  $\text{cc}(G_n) \geq n + 2$ . Now, we show that  $G_n$  has a unique clique covering containing exactly  $n + 2$  cliques. Let  $\mathcal{C}$  be a clique covering of  $G_n$  consisting of  $n + 2$  cliques. Assume that the clique  $C_i \in \mathcal{C}$  covers the edge  $x_i y_i$ , for  $1 \leq i \leq n$ , and the cliques  $C_{n+1} \in \mathcal{C}$  and  $C_{n+2} \in \mathcal{C}$  cover the edges  $y_0 y_1$  and  $x_0 x_1$ , respectively. Note that  $C_{n+2} \subseteq \{x_0\} \cup X$  and  $x_0 \notin \cup_{i=1}^{n+1} C_i$ . Therefore,  $C_{n+2} = \{x_0\} \cup X$ . Similarly,  $C_{n+1} = \{y_0\} \cup Y$ . Also, we have  $x_j, y_j \notin C_i$ , for every  $1 \leq i \neq j \leq n$ . Thus,  $C_i = \{x_i, y_i\} \cup Z$ ,  $1 \leq i \leq n$ . Hence, the clique covering  $\mathcal{C} = \{C_i : 1 \leq i \leq n + 2\}$  is the unique clique covering of  $G_n$  with  $n + 2$  cliques and then  $\text{cc}(G_n) = n + 2$ . Consequently,

$$\text{scc}'(G_n) = \sum_{C \in \mathcal{C}} |C| = n(n + 2) + 2(n + 1) = n^2 + 4n + 2.$$

On the other hand, the  $n + 4$  cliques  $\{x_0\} \cup X$ ,  $\{y_0\} \cup Y$ ,  $X \cup Z$ ,  $Y \cup Z$  and  $\{x_i, y_i\}$ ,

$1 \leq i \leq n$ , form a clique covering  $\mathcal{C}'$  and thus,

$$\text{scc}(G_n) \leq \sum_{C \in \mathcal{C}'} |C| = 2(n+1) + 2(2n) + 2n = 8n + 2.$$

Hence, the families of the optimum clique coverings achieving  $\text{cc}(G_n)$  and  $\text{scc}(G_n)$  are disjoint and  $\text{scc}'(G_n)/\text{scc}(G_n)$  tends to infinity.  $\square$

In the following, we prove some relations between  $\text{scc}(G)$ ,  $\text{scp}(G)$  and  $\text{cp}(G)$ .

**Theorem 2.** *If  $G$  is a graph with  $m$  edges and  $\omega(G)$  is the clique number of  $G$ , then*

- i)  $\frac{2m}{\omega(G) - 1} \leq \text{scc}(G) \leq \text{scp}(G) \leq 2m,$
- ii)  $\frac{\text{scp}^2(G)}{2m + \text{scp}(G)} \leq \text{cp}(G).$

*Also, in all relations, the equalities hold for the triangle-free graphs.*

*Proof.* i) Since the collection of all edges of  $G$  is a clique partition for  $G$ , we have  $\text{scc}(G) \leq \text{scp}(G) \leq 2m$ . Now, suppose that  $\mathcal{C}$  is a clique covering of  $G$  such that  $\sum_{C \in \mathcal{C}} |C| = \text{scc}(G)$ . Clearly  $m \leq \sum_{C \in \mathcal{C}} \binom{|C|}{2}$ . Hence,

$$2m \leq \sum_{C \in \mathcal{C}} |C|^2 - \text{scc}(G) \leq (\omega(G) - 1) \text{scc}(G).$$

ii) Let  $\text{cp}(G) = t$  and  $\{C_1, \dots, C_t\}$  be a clique partition of  $G$ . Then,  $m = \sum_{i=1}^t \binom{|C_i|}{2}$ . Thus,

$$\begin{aligned} 2m &= \sum_{i=1}^t |C_i|^2 - \sum_{i=1}^t |C_i| \\ &\geq \frac{1}{t} \left( \sum_{i=1}^t |C_i| \right)^2 - \sum_{i=1}^t |C_i| \\ &\geq \frac{1}{t} \text{scp}^2(G) - \text{scp}(G), \end{aligned}$$

where the second inequality is due to Cauchy-Schwarz inequality and the last inequality holds because the function  $f(x) = \frac{1}{t}x^2 - x$  is increasing for  $x \geq \frac{t}{2}$  and clearly  $\text{scp}(G) \geq \text{cp}(G) = t$ .  $\square$

For a vertex  $u \in V(G)$ , let  $N_G(u)$  denotes the set of all neighbours of  $u$  in  $G$  and let  $\overline{G}$  stand for the complement of  $G$ . Moreover, let  $\Delta(G)$  be the maximum degree of  $G$ . Alon in [1] proved that if  $G$  is a graph on  $n$  vertices and  $\Delta(\overline{G}) = d$ , then  $\text{cc}(G) = O(d^2 \log n)$ . In the following, modifying the idea of Alon, we establish an upper bound for  $\text{scc}(G)$ .

**Theorem 3.** *If  $G$  is a graph on  $n$  vertices with no isolated vertex and  $\Delta(\overline{G}) = d - 1$ , then*

$$\text{scc}(G) \leq (e^2 + 1)nd \left\lceil \ln \left( \frac{n-1}{d-1} \right) \right\rceil. \quad (1)$$

*Proof.* Let  $0 < p < 1$  be a fixed number and let  $S$  be a random subset of  $V(G)$  defined by choosing every vertex  $u$  independently with probability  $p$ . For every vertex  $u \in S$ , if there exists a non-neighbour of  $u$  in  $S$ , then remove  $u$  from  $S$ . The resulting set is a clique of  $G$ . Repeat this procedure  $t$  times, independently, to get  $t$  cliques  $C_1, C_2, \dots, C_t$  of  $G$ .

Let  $F$  be the set of all the edges which are not covered by the cliques  $C_1, \dots, C_t$ . For every edge  $uv$ , using inequality  $(1 - \alpha) \leq e^{-\alpha}$ , we have

$$\Pr(uv \in F) = (1 - p^2(1 - p))^{|N_{\overline{G}}(u) \cup N_{\overline{G}}(v)|} \leq (1 - p^2(1 - p))^{2(d-1)t} \leq e^{-tp^2(1-p)^{2(d-1)}}.$$

The cliques  $C_1, \dots, C_t$  along with all edges in  $F$  comprise a clique covering of  $G$ . Hence,

$$\begin{aligned} \text{scc}(G) &\leq \mathbf{E} \left( \sum_{i=1}^t |C_i| + 2|F| \right) \\ &\leq npt + 2 \binom{n}{2} e^{-tp^2(1-p)^{2(d-1)}}. \end{aligned}$$

Now, set  $p := 1/d$ . Since  $(1 - 1/d)^{d-1} \geq 1/e$ , we have

$$\text{scc}(G) \leq \frac{nt}{d} + n(n-1)e^{-td^{-2}e^{-2}}.$$

Finally, by setting  $t := \lceil e^2 d^2 \ln(\frac{n-1}{d-1}) \rceil > 0$ , we have

$$\begin{aligned} \text{scc}(G) &\leq \frac{n(e^2 d^2 \ln(\frac{n-1}{d-1}) + 1)}{d} + n(d-1) \\ &\leq nd \left\lceil \ln \left( \frac{n-1}{d-1} \right) \right\rceil \left( e^2 + \frac{1}{\lceil \ln(\frac{n-1}{d-1}) \rceil} \right) \\ &\leq nd \left\lceil \ln \left( \frac{n-1}{d-1} \right) \right\rceil (e^2 + 1). \end{aligned}$$

□

The upper bound in (1) gives rise to the question that for positive integers  $n, d$ , how large can be the sigma clique cover number of an  $n$ -vertex graph where the maximum degree of its complement is  $d - 1$ . A first candidate for graphs with large scc is the family of complete multipartite graphs.

For positive integers  $n, k$ , an *orthogonal array*  $\text{OA}(n, k)$  is an  $n^2 \times k$  array of elements in  $\{1, \dots, n\}$ , such that in every two columns each ordered pair  $(i, j)$ ,  $1 \leq i, j \leq n$ , appears exactly once.

**Theorem 4.** *For positive integers  $n, d$  with  $n \geq 2d$ , let  $G$  be a complete multipartite graph on  $n$  vertices with at least two parts of size  $d$  and the other parts of size at most  $d$ . Then,  $\Delta(\overline{G}) = d - 1$  and  $\text{scc}(G) \geq nd$ . Moreover, if  $d$  is a prime power and  $n \leq d(d+1)$ , then  $\text{scc}(G) = \text{scp}(G) = nd$ .*

*Proof.* Let  $\mathcal{C}$  be a clique covering for  $G$ . For every vertex  $u$ ,  $N_G(u)$  contains a stable set (a set of pairwise nonadjacent vertices) of size  $d$ . Therefore,  $u$  is contained in at least  $d$  cliques of  $\mathcal{C}$ , i.e. the valency of  $u$ ,  $\mathcal{V}_{\mathcal{C}}(u)$  is at least  $d$ . Thus,  $\text{scc}(G) \geq nd$ .

Now, let  $d$  be a prime power. It is known that there exists an orthogonal array  $\text{OA}(d, d+1)$ . Let  $k = d + 1$  and denote the  $i$ th row of the orthogonal array by  $a_{i1}, a_{i2}, \dots, a_{ik}$ . Also, let  $H$  be a complete  $k$ -partite graph on  $d(d+1)$  vertices with the parts  $V_1, \dots, V_k$ , where  $V_j = \{v_{j1}, \dots, v_{jd}\}$ , for  $1 \leq j \leq k$ . For each  $i \in \{1, \dots, d^2\}$ , the set  $C_i := \{v_{1a_{i1}}, v_{2a_{i2}}, \dots, v_{ka_{ik}}\}$  is a clique of  $H$ . Since in every two columns of  $\text{OA}$ , each ordered pair  $(i, j)$ ,  $1 \leq i, j \leq d$ , appears exactly once, the collection  $\mathcal{C} := \{C_i : 1 \leq i \leq d^2\}$  forms a clique partition for  $H$ . Moreover, for every vertex  $u \in V(H)$ ,  $\mathcal{V}_{\mathcal{C}}(u) = d$ . On the other hand,  $G$  is an induced subgraph of  $H$ . Thus, the collection  $\mathcal{C}' := \{C_i \cap V(G) : 1 \leq i \leq d^2\}$  is a clique partition of  $G$  and for every vertex  $u \in V(G)$ ,  $\mathcal{V}_{\mathcal{C}'}(u)$  is at most  $d$ . Hence,  $\text{scc}(G) \leq \text{scp}(G) \leq nd$ .  $\square$

For positive integers  $t, d$ , let us denote the complete  $t$ -partite graph with each part of size  $d$  by  $K_t(d)$ . Theorem 3 asserts that  $\text{scc}(K_t(d)) \leq cd^2t \log t$ , for some constant  $c$ . Although Theorem 4 says that  $\text{scc}(K_t(d)) = d^2t$  when  $t \leq (d+1)$  and  $d$  is a prime power, we believe that  $\text{scc}(K_t(d))$  is much larger when  $t$  is sufficiently large. This leads us to the following conjecture.

**Conjecture 5.** There exists a function  $f$  and a constant  $c$ , such that for every positive integers  $t$  and  $d$ , if  $t \geq f(d)$ , then  $\text{scc}(K_t(d)) \geq cd^2t \log t$ .

In fact, if Conjecture 5 is correct, then the upper bound in (1) is best possible up to a constant factor, at least for sufficiently large  $n$ . In the following section, we will prove that Conjecture 5 is true for  $d = 2$ .

### 3 Cocktail Party Graphs

In this section, we investigate the sigma clique cover number of the Cocktail party graph  $K_t(2)$ . Given a positive integer  $t$ , the Cocktail party graph  $K_t(2)$  is obtained from the complete graph  $K_{2t}$  with the vertex set  $\{x_1, \dots, x_t\} \cup \{y_1, \dots, y_t\}$  by removing all the edges  $x_i y_i$ ,  $1 \leq i \leq t$ .

Various clique covering parameters of the Cocktail party graphs have been studied in the literature. In 1977, Orlin [12] asked about asymptotic behaviour of  $\text{cc}(K_t(2))$ , with this motivation that it arises in an optimization problem in Boolean functions theory. He also conjectured that  $\text{cp}(K_t(2)) \sim t$ . Gregory et al. [8] proved that for  $t \geq 4$ ,  $\text{cp}(K_t(2)) \geq 2t$  and for large enough  $t$ ,  $\text{cp}(K_t(2)) \leq 2t \log \log 2t$ . The problem that  $\text{cp}(K_t(2)) \sim 2t$

is still an open problem. Moreover, Gregory and Pullman [9], by applying a Sperner-type theorem of Bollobás and Schönheim on set systems, proved that for every integer  $t$ ,  $\text{cc}(K_t(2)) = \sigma(t)$ , where

$$\sigma(t) = \min \left\{ k : t \leq \binom{k-1}{\lceil k/2 \rceil} \right\}.$$

Furthermore, the authors in [5], using the pairwise balanced designs, have proved that  $\text{scp}(K_t(2)) \sim (2t)^{3/2}$ .

Here, using the following well-known theorem by Bollobás, we prove a lower bound for the sigma clique cover number of  $K_t(2)$  which determines the asymptotic behaviour of  $\text{scc}(K_t(2))$  and implies that Conjecture 5 is true for  $d = 2$ .

**Bollobás' Theorem.** [2] *Let  $A_1, \dots, A_t$  be some sets of size  $a_1, \dots, a_t$ , respectively and  $B_1, \dots, B_t$  be some sets of size  $b_1, \dots, b_t$ , respectively, such that  $A_i \cap B_j = \emptyset$  if and only if  $i = j$ . Then*

$$\sum_{i=1}^t \binom{a_i + b_i}{a_i}^{-1} \leq 1.$$

**Theorem 6.** *Let  $K_t(2)$  be the Cocktail party graph on  $2t$  vertices. Then*

$$t\delta(t) \leq \text{scc}(K_t(2)) \leq t\sigma(t),$$

where  $\sigma(t)$  is defined as above and  $\delta(t) = \min \left\{ k-1 : t \leq \binom{k}{\lceil k/2 \rceil} \right\}$ .

*Proof.* Since  $\text{cc}(K_t(2)) = \sigma(t)$  and every clique in  $K_t(2)$  is of size at most  $t$ , we have  $\text{scc}(K_t(2)) \leq t\sigma(t)$ .

For the lower bound, assume that  $\{C_1, \dots, C_k\}$  is an arbitrary clique covering for  $K_t(2)$ . For every  $i \in \{1, \dots, t\}$ , define

$$A_i = \{a : x_i \in C_a\}, \quad B_i = \{a : y_i \in C_a\}.$$

Also, let  $a_i = |A_i|, b_i = |B_i|$  and  $c_i = a_i + b_i$ . Then for every  $i \neq j$ , there exists a clique containing the edge  $x_i y_j$ . Hence,  $A_i \cap B_j \neq \emptyset$ . Moreover, since no clique contains both vertices  $x_i$  and  $y_i$ , we have  $A_i \cap B_i = \emptyset$ .

Therefore, by Bollobás' theorem, we have

$$\sum_{i=1}^t \binom{a_i + b_i}{a_i}^{-1} \leq 1.$$

For every integer  $m$ , let  $f(m) = \binom{m}{\lceil m/2 \rceil}^{-1}$  and  $f(x)$  be the linear extension of  $f(m)$  in  $\mathbb{R}^+$ . Since  $f$  is non-increasing and convex, by Jensen inequality, we have

$$f\left(\left\lceil \frac{1}{t} \sum_{i=1}^t c_i \right\rceil\right) \leq f\left(\frac{1}{t} \sum_{i=1}^t c_i\right) \leq \frac{1}{t} \sum_{i=1}^t \binom{c_i}{\lceil c_i/2 \rceil}^{-1} \leq \frac{1}{t} \sum_{i=1}^t \binom{a_i + b_i}{a_i}^{-1} \leq \frac{1}{t}.$$

Thus,  $\left(\left\lceil \frac{1}{t} \sum_{i=1}^t c_i \right\rceil\right) \geq t$ . Therefore,

$$\delta(t) \leq \left\lceil \frac{1}{t} \sum_{i=1}^t c_i \right\rceil - 1 \leq \frac{1}{t} \sum_{i=1}^t c_i = \frac{1}{t} \sum_{a=1}^k |C_a|.$$

Consequently,  $t\delta(t) \leq \text{scc}(K_t(2))$ . □

Theorem 6 along with the approximation  $\binom{2n}{n} \sim 2^{2n}/\sqrt{\pi n}$  yields the following corollary which proves Conjecture 5 for  $d = 2$ .

**Corollary 7.** *For every integer  $t$ ,  $\text{scc}(K_t(2)) \sim t \log t$ .*

## 4 Concluding Remarks

In previous section, by considering a clique covering as a set system and applying Bollobás' theorem, we proved Conjecture 5 for  $d = 2$ . In this point of view, this conjecture can be restated as an interesting set system problem and thus it can be viewed as a generalization of Bollobás' theorem, as follows.

**Conjecture 8.** Let  $d \geq 2$ ,  $t \geq 1$  and  $\mathcal{F} = \{(A_i^1, A_i^2, \dots, A_i^d) : 1 \leq i \leq t\}$  such that  $A_i^j$  is a set of size  $k_{ij}$  and  $A_i^j \cap A_{i'}^{j'} = \emptyset$  if and only if  $i = i'$  and  $j \neq j'$ . Then, there exists a function  $f$  and a constant  $c$ , such that for every  $t \geq f(d)$ ,

$$\sum_{i,j} k_{ij} \geq cd^2 t \log t.$$

Note that Conjecture 8 is true for  $d = 2$ , due to Bollobás' theorem.

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